Hamiltonian and its solution

So, before t_0 we have our field, reference, all bath modes and the interaction with the system. So, the field is not affected by the system at t_0 and the bath is still free, provided we write

$$\hat{H}_{SB} = \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k + \sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)$$

Then we can write the time operator (eq. 1) as a function of $\hat{\sigma}_-(t)$.

By operator order, transition between different states of $\hat{H}$ not the same as original $\hat{H}$ because of interaction with the environment. Thus, we define the operator $\hat{\sigma}_-$ as a function of time.

Now, we can write the following relations. (Eq. 2) can be derived from eq. 1 and eq. 3.

$$\hat{\sigma}_-(t) = \hat{\sigma}_- e^{-i\omega_0 t}$$

$$\hat{\sigma}_+(t) = \hat{\sigma}_+ e^{i\omega_0 t}$$

Now, we may write

$$\hat{\sigma}_z(t) = \hat{\sigma}_z^2 e^{-i\omega_0 t} e^{i\omega_0 t} = \hat{\sigma}_z$$

So, the system, or atom, is affected by the bath, perhaps, getting hot. Not exactly the same as original, however, it is in the form of a function of time.

2. Evolution of the Reservoir operators

2.1 Heisenberg Picture: Evolution of the reservoir operators

Coupled system:

(1) Interacting system: $\hat{H}_{SB}$

(2) Free bath: $\sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k$

$$\hat{H} = \hbar\omega_0(\hat{\sigma}_z + \hat{\mathbb{1}})$$

Particular states:

- Field states: $|n\rangle$ operator of $\hat{b}^\dagger$ $\hat{b}$ (Fock states) (stationary)
- Coherent states: $|\alpha\rangle$ eigenvalues, not eigenvalues (non-stationary) $\hat{b}|\alpha\rangle = |\alpha\rangle\alpha$, $\langle\alpha|\hat{b}^\dagger = \langle\alpha|\alpha^*$

$$\langle\alpha|\alpha\rangle = 1 \quad \langle\alpha|\alpha\rangle = 1 \quad \langle\alpha|\alpha\rangle = 1$$

Coherent operator: Line of transition between states of the system, or bath, not generated by $\hat{\sigma}_+$ and $\hat{\sigma}_-$, the eigenvalue for of the state operator, we can compute as

$$\frac{d\hat{\sigma}_-}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{\sigma}_-]$$

$$= \frac{i}{\hbar}\left(\sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k \hat{\sigma}_- - \hat{\sigma}_- \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right) + \frac{i}{\hbar}\left(\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\hat{\sigma}_- - \hat{\sigma}_-\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\right)$$

This is nothing but the form we found in Section 2 and 3 of the system.

Nothing $\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\hat{\sigma}_- = \hat{\sigma}_-^2 = 0$

$$\frac{d\hat{\sigma}_-}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{\sigma}_-]$$

$$= \frac{i}{\hbar}\left(\sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k \hat{\sigma}_- - \hat{\sigma}_- \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right) + \frac{i}{\hbar}\left(\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\hat{\sigma}_- - \hat{\sigma}_-\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\right)$$

(2) The result is independent of the initial state of the system.

2.2 Evolution of a free bath

Then we have general equation:

$$\frac{d\hat{\sigma}_-}{dt} = \frac{i}{\hbar}\left(\sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k \hat{\sigma}_- - \hat{\sigma}_- \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right) + \frac{i}{\hbar}\left(\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\hat{\sigma}_- - \hat{\sigma}_-\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\right)$$

(1) Bosons, $\hat{b}$: $[\hat{b}, \hat{b}^\dagger] = 1$

Case: $\left[\hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = \hbar\omega_k \hat{b}_k$ (non-zero commutator)

(2) Bosonic Group

Probability: $P_n(\omega) = T_n(\omega - \omega_0)$

Case: $\left[\omega_k \hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = \hbar\omega_k \hat{b}_k$ (non-zero commutator)

(3) Bosonic Group

Probability: $P_n(\omega) = T_n(\omega)$

Case: $\left[\omega_k \hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = 0$ (non-zero commutator)

N.B. Find how a pure mixture, or non-excitation, is generated in one of the reservoir modes of the bath. Therefore, the reservoir, or bath, of the system is generated by a subsequent number of the bath and we find a distribution.

$$\langle\hat{\sigma}(t)\rangle = \text{Tr}(\hat{\sigma}\hat{\sigma}_-) = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \langle n|\hat{\sigma}_-|n\rangle$$

2.3 Evolution of a coherent state

Then we have general equation:

$$\frac{d\hat{\sigma}_-}{dt} = \frac{i}{\hbar}\left(\sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k \hat{\sigma}_- - \hat{\sigma}_- \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right) + \frac{i}{\hbar}\left(\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\hat{\sigma}_- - \hat{\sigma}_-\sum_k \hbar\left(\hat{b}_k^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{b}_k\right)\right)$$

(1) Bosons, $\hat{b}$: $[\hat{b}, \hat{b}^\dagger] = 1$

Case: $\left[\hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = \hbar\omega_k \hat{b}_k$ (non-zero commutator)

(2) Bosonic Group

Probability: $P_n(\omega) = T_n(\omega - \omega_0)$

Case: $\left[\omega_k \hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = \hbar\omega_k \hat{b}_k$ (non-zero commutator)

N.B. Find how a pure mixture, or non-excitation, is generated in one of the reservoir modes of the bath. Therefore, the reservoir, or bath, of the system is generated by a subsequent number of the bath and we find a distribution.

$$\langle\hat{\sigma}(t)\rangle = \text{Tr}(\hat{\sigma}\hat{\sigma}_-) = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \langle n|\hat{\sigma}_-|n\rangle$$

2.4 Evolution of a coherent state

Then we without reviewing, we find the same state that is a single, pure operator. $\hat{b}_k = \hat{b}^\dagger$. The $\hat{\sigma}_-$ is unaffected. So, we find the system is a real bath, at non-temperature.

$$\langle\alpha|\hat{\sigma}_-|\alpha\rangle$$

We find the pressure:

$$\langle\alpha|\hat{\sigma}_-|\alpha\rangle$$

(1) Bosons, $\hat{b}$: $[\hat{b}, \hat{b}^\dagger] = 1$

$$\left[\hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = \hbar\omega_k \hat{b}_k$$

(2) Bosonic Group

$$P_n(\omega) = T_n(\omega - \omega_0)$$

$$\left[\omega_k \hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = \hbar\omega_k \hat{b}_k$$

(3) Bosonic Group

$$P_n(\omega) = T_n(\omega)$$

$$\left[\omega_k \hat{b}_k, \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k\right] = 0$$

(4) The coherent state is not affected by a pure mixture.

Remark: at the first time, we are affected by the $\hat{\sigma}_+$ and $\hat{\sigma}_-$ operators. So, the system is in a state with coherent states that is affected by the pure operator $\hat{\sigma}_-$.

(1) The first state, $|\alpha\rangle$ is affected by the coherent state of a pure mixture, or non-excitation. We may believe that the pure state operator has been created in the bath, right after the $\hat{\sigma}_-$ is affected by the operator of $\hat{\mathbb{1}}$. We should however, remember that the Hamiltonian, $\hat{H}$, operator is a sum of measurement, therefore, we can write a $\hat{H}$ operator as not an operator, but a function. Thus, the operator, $\hat{\sigma}_-$, is a function of time. Thus, the $\hat{\sigma}_-$ operator is affected by the pure mixture, or non-excitation, that is, the particle. Thus, the $\hat{\sigma}_-$

Open Quantum Systems

Laurent Sanchez-Palencia

Center for Theoretical Physics

Ecole Polytechnique, CNRS, Institut Polytechnique de Paris

F-91128 Palaiseau, France

Programme

1. Introduction to open quantum systems
   - 1.1 Position of the problem
   - 1.2 Simple approaches
   - 1.3 The density operator (reminder ... or not)
2. Master equation : The statistical physics approach
   - 2.1 Formal derivation of the master equation
   - 2.2 Statistical equilibrium : The damped harmonic oscillator
   - 2.3 Driven systems : The Optical Bloch equations
3. The quantum information approach
   - 3.1 Kraus operators
   - 3.2 Lindblad form of the master equation
4. Decoherence and its control
   - 4.1 Decoherence of the harmonic oscillator
   - 4.2 Pointer states
   - 4.3 Discussion

Summary

Lecture 1

Wave function approaches (perturbation theory and Weisskopf-Wigner) allowed us to identify the main concepts at the heart of the dynamics of open quantum systems :

- Separation of time scales due to a weak coupling between S and B
- Onset of a Markov process (loss of memory)
- Relaxation of the initial state (crucial role of the continuum of states generated by a bath with a huge number of degrees of freedom)

The density operator, which generalizes the notion of « state » of a quantum system, is the appropriate framework to describe open quantum systems

- Two equivalent formulations (S may be in a set of possible kets with a certain probability distribution ; S is entangled with B and its density operator is found by tracing out the degrees of freedom of B)
- The dynamics of an isolated system is governed by the quantum Liouville equation

$$i\hbar \frac{d\hat{\rho}_S}{dt} = \left[\hat{H}_S(t), \hat{\rho}_S\right]$$

Summary

Lecture 2

The dynamics of an open quantum system is governed by a master equation of the form

$$\frac{d\hat{\rho}_S}{dt} = \mathscr{L}\left[\hat{\rho}_S(t)\right] = \frac{1}{i\hbar}\left[\hat{H}_S{}', \hat{\rho}_S(t)\right] + \mathscr{L}'\left[\hat{\rho}_S(t)\right]$$

where $\mathscr{L}\left[\hat{\rho}_S(t)\right]$ is the Liouvillian, $\hat{H}_S{}'$ is the Hamiltonian of the system, possibly renormalized by its interaction with the bath, and $\mathscr{L}'\left[\hat{\rho}_S(t)\right]$ induces a non-unitary evolution.

Working within the framework of the density operator, assuming that the bath is weakly affected by its interaction with the system (while the system may be strongly affected), and using the hierarchy of time scales, we have derived a general form of the master equation,

$$\frac{d\hat{\rho}_S}{dt} = \frac{1}{i\hbar}\left[\hat{H}_S(t), \hat{\rho}_S(t)\right] - \frac{1}{\hbar^2}\int_0^\infty d\tau \ \mathrm{Tr}_B\left[\hat{H}_I(t), \left[\hat{H}_I(t-\tau), \hat{\rho}_S(t) \otimes \hat{\rho}_B(0)\right]\right]$$

Summary

Lecture 2 (continued)

We have discussed two applications of the master equation, where a rather simple form can be derived :

Damped harmonic oscillator :

- The dynamics of the populations is a simple classical rate equation, involving terms, which are easily interpreted as spontaneous emission, and stimulated emission and absorption (at finite temperature)
- Emergence of a statistical equilibrium (Boltzmann distribution). If the bath is at thermal equilibrium (Bose-Einstein distribution), the temperature of the system equilibrates to that of the bath.
- Dynamical damping of oscillations

Atom in a laser field (driven system, governed by the optical Bloch equations)

- Dynamics of an atom at rest (damped Rabi oscillations, relaxation of populations and coherences at a typical rate larger than $\Gamma/2$)
- Dynamics of a moving atom and manipulation of atoms by laser (radiation pressure and dipole force)

Summary

Lecture 3

The evolution of a system S coupled to another system B (whatever they are) can always be written as

$$\hat{\rho}(t) = \sum_{0 \leqslant k < K} \hat{E}_k \hat{\rho}(0) \hat{E}_k^\dagger$$

where the $\hat{E}_k$ are Kraus operators and fulfill the completion relation $\sum_{0 \leqslant k < K} \hat{E}_k^\dagger \hat{E}_k = 1$.

This is the most general form of a quantum map $\hat{\rho}(0) \rightarrow \hat{\rho}(t) = \mathscr{K}[\hat{\rho}(0)]$, which preserves the density operator character of $\hat{\rho}(t)$ (Hermitian, non-negative, of trace unity).

The number K of Kraus operators may be set to

$K \leqslant \dim(\mathscr{H}_\mathrm{B})$ for B initially in a pure state

$K \leqslant \dim(\mathscr{H}_\mathrm{B})^2$ for B initially in an arbitrary state

$K \leqslant \dim(\mathscr{H}_\mathrm{S})^2$: most compact form (interpretation as the coupling to a fictious bath)

Summary

Lecture 3 (continued)

A Kraus operator represents the probability amplitude that B jumps from a state to another :

$$P_k = \mathrm{Tr}_S\left(\hat{E}_k \hat{\rho}(0) \hat{E}_k^\dagger\right)$$

A jump of the state of B is accompanied by a jump of the state of S.

A Kraus operator describes the evolution of S conditional to the measurement of the state of B :

$$\hat{\rho}'_{S|k} = \frac{\hat{E}_k \hat{\rho}(0) \hat{E}_k^\dagger}{\mathrm{Tr}_S\left(\hat{E}_k \hat{\rho}(0) \hat{E}_k^\dagger\right)}$$

The evolution of S is the average of its evolution conditional to the result of the measurement of the state of B :

$$\hat{\rho}_S = \sum_k P_k \times \hat{\rho}'_{S|k}$$

Summary

Lecture 4

The most general form of a master equation is the Lindblad form

$$\frac{d\hat{\rho}_S}{dt} = \frac{1}{i\hbar}\left[\hat{H}_S{}', \hat{\rho}_S\right] + \mathscr{L}'\left[\hat{\rho}_S\right]$$

with $$\mathscr{L}'(\hat{\rho}_S) = \sum_{1 \leqslant k < K} \left(\hat{L}_k \hat{\rho}_S \hat{L}_k^\dagger - \frac{1}{2} \hat{\rho}_S \hat{L}_k^\dagger \hat{L}_k - \frac{1}{2} \hat{L}_k^\dagger \hat{L}_k \hat{\rho}_S \right)$$

They are found by writting

$$\left| \begin{array}{l} \hat{E}_{k=0} \simeq 1 - \left(\frac{1}{2} \sum_{1 \leqslant k < K} \hat{L}_k^\dagger \hat{L}_k - \frac{1}{i\hbar} \hat{H} \right) \Delta t \\ \\ \hat{E}_{k>0} \simeq \hat{L}_k \sqrt{\Delta t} \end{array} \right.$$

This representation permits a stroboscopic evolution of the density operator of S (Markov hypothesis, stochastic wavefunctions) : Isolated evolution interrupted by jumps induced by measurements of the sate of B)

Summary

Lecture 4 (continued)

Reintroducing the states of the B, the stochastic wave function approach allows to efficiently describe decoherence.

For a harmonic oscillator, the Fock states are subjected to decoherence. In contrast, the coherent states evolve as pure coherent states (although with a damping of the parameter α).

A state that is protected against decoherence (ie a state that evolves as a pure state, non entangled with the state of the bath) is called a pointer state. They play a central role in the theory of quantum measurement.

To go beyond these lectures, strategies to fight against decoherence :

- Design of the bath (quantum cavity electrodynamics, trapped ions)
- Design the coupling (quantum Zeno effect)
- Reconstruct coherence by measuring the environment (quantum eraser and feedback)