Open Quantum Systems

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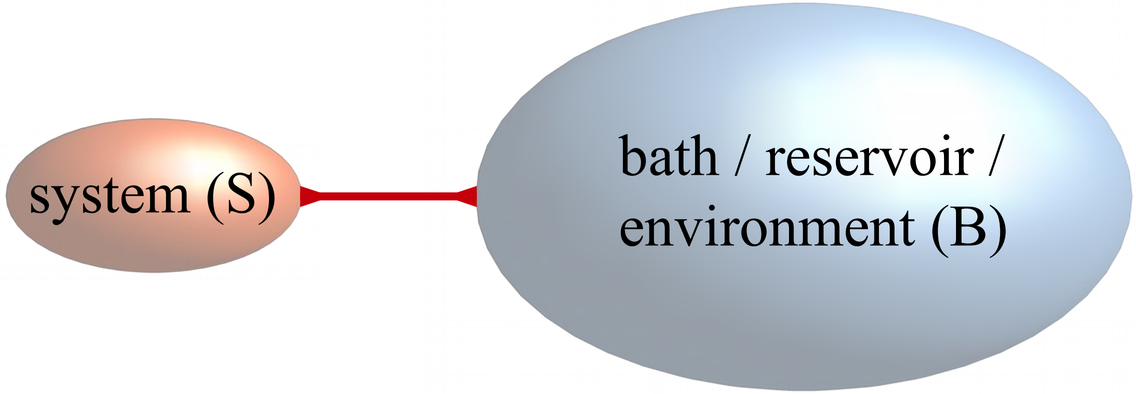
Open Quantum Systems

Here, open = non-isolated
(ie « open » or « closed » in
the thermodynamic sense)



Open systems are ubiquitous

Uncontrolled : atom in vacuum, electrons in a solid, nanomechanical systems, …

Controlled : manipulation, engineering, measurements

Formal : A small part of a larger system

Many physical consequences

Spontaneous emission of individual atoms

Emergence of statistical physics

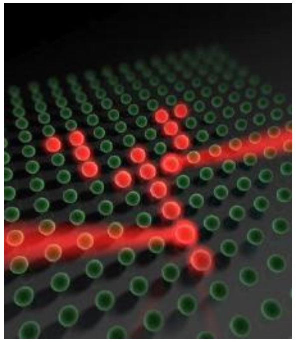
Decoherence, classical/quantum frontier

Theory of quantum measurement

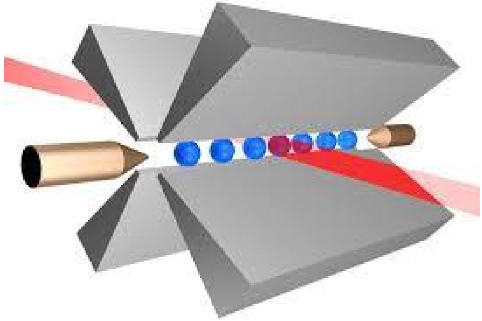
Relaxation (NMR, 1950's ; Optical pumping, 1960's ...)

Control of individual quantum systems

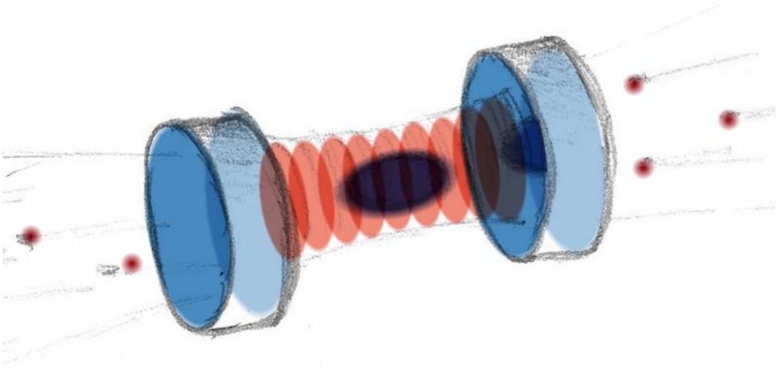
Open Quantum Systems



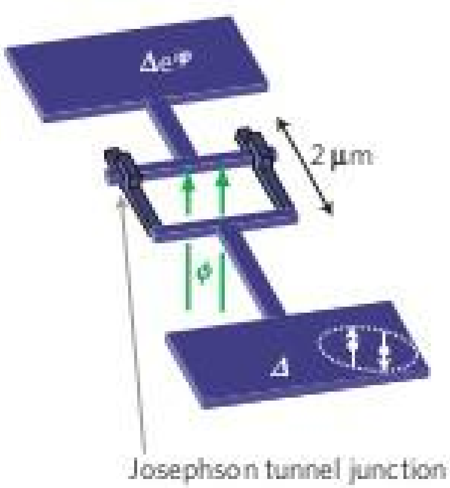
ultracold atoms



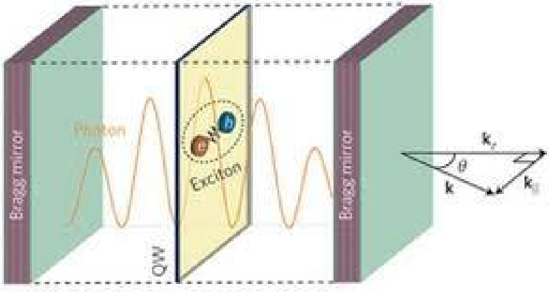
artificial ion crystals



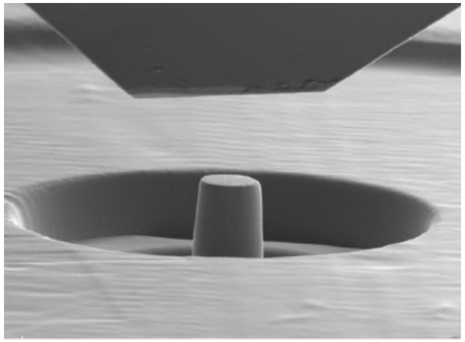
cavity QED



superconducting circuits



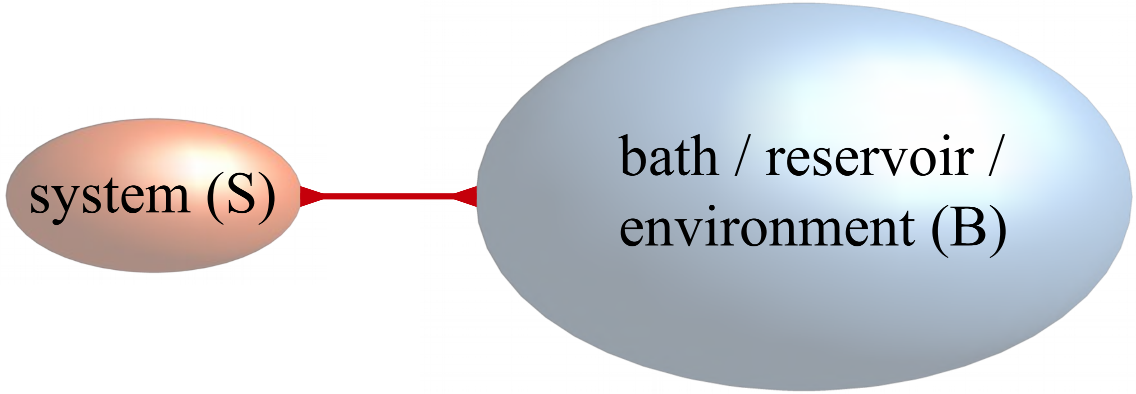
cavity polaritons



nanomechanical devices

Open Quantum Systems

How to describe the dynamics of an open quantum system ?



Open Quantum Systems

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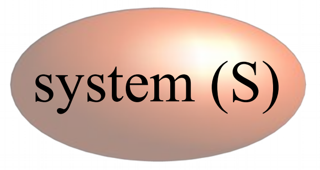


Wavefunction approach

Schrödinger equation

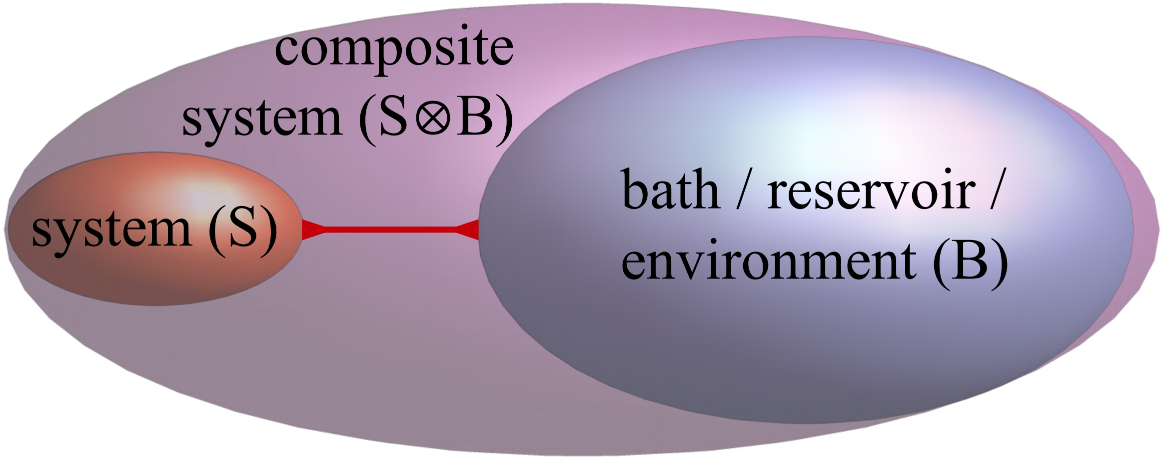
$$i\hbar \frac{d|\Psi_{\mathrm{S}}\rangle}{dt} = \hat{H}_{\mathrm{S}}(t)|\Psi_{\mathrm{S}}\rangle$$

Superpositions, unitarity, reversibility

$$|\Psi_{\mathrm{S}}(t)\rangle = \hat{U}_{\mathrm{S}}(t)|\Psi_{\mathrm{S}}(0)\rangle$$

$$|\Psi_{\mathrm{S}}(0)\rangle = \hat{U}_{\mathrm{S}}^{\dagger}(t)|\Psi_{\mathrm{S}}(t)\rangle$$

Open Quantum Systems



How to describe the dynamics of an open quantum system ?

Wavefunction approach

Schrödinger equation

$$i\hbar \frac{d|\Psi_{\mathrm{S\otimes B}}\rangle}{dt} = \hat{H}_{\mathrm{S\otimes B}}(t)|\Psi_{\mathrm{S\otimes B}}\rangle$$

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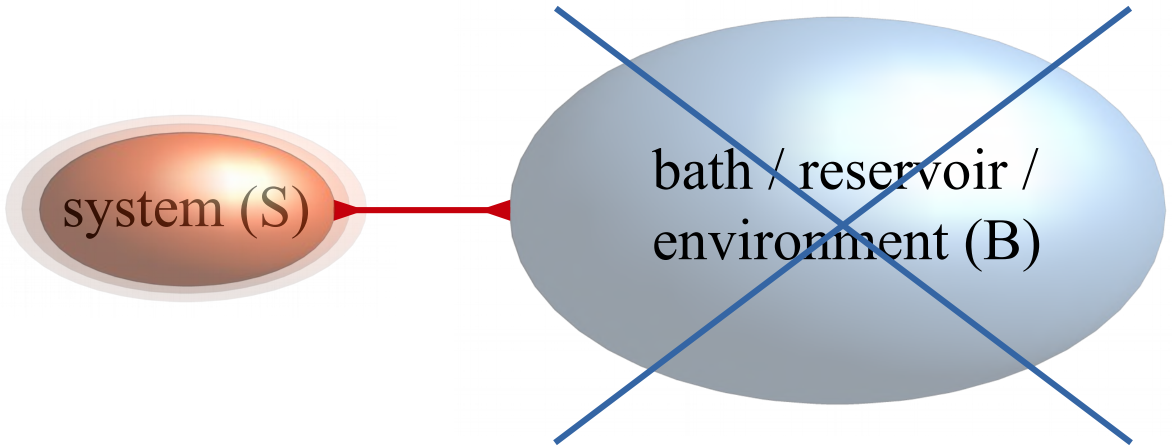
$$|\Psi_{\mathrm{S\otimes B}}(t)\rangle = \hat{U}_{\mathrm{S\otimes B}}(t)|\Psi_{\mathrm{S\otimes B}}(0)\rangle$$

$$|\Psi_{\mathrm{S\otimes B}}(0)\rangle = \hat{U}^{\dagger}_{\mathrm{S\otimes B}}(t)|\Psi_{\mathrm{S\otimes B}}(t)\rangle$$

⇒ Keeps all about S⊗B, but information on S is hidden

Open Quantum Systems

How to describe the dynamics of an open quantum system ?



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⇒ Keeps all about S⊗B, but information on S is hidden

Description by the density operator

Master equation

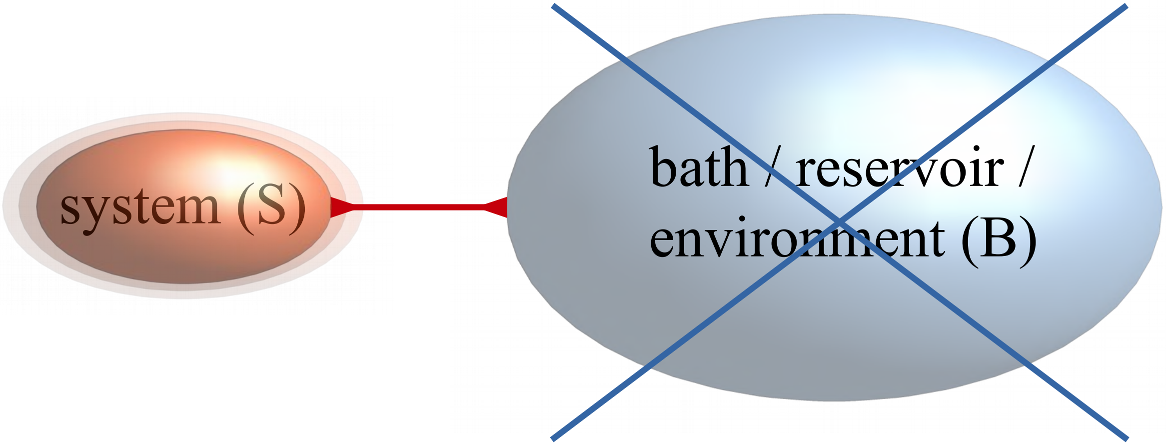
$$\frac{d\hat{\rho}_S}{dt}=\mathscr{L}[\hat{\rho}_S(t)]=\frac{1}{i\hbar}[\hat{H}_S,\hat{\rho}_S(t)]+\mathscr{L}'[\hat{\rho}_S(t)]$$

Non-unitary map, irreversibility/relaxation, loss of coherence

$$\hat{\rho}_S(t)=K_t[\hat{\rho}_S(0)]$$

⇒ Relevant formalism for an open system (information on B is traced out)

Scope of these Lectures



Work within the scope of the density operator

How to derive a master equation ?

Not easy : How to deal with the information leak towards B, memory effects ? Not so clear that B can be traced out …

What is the physical content of the master equation (emergence of statistical physics, manipulation, control …) ?

How to relate the trajectory of an individual quantum system to the dynamics of the density operator ?

Probable Programme

1. Introduction to open quantum systems
   - 1.1 Position of the problem
   - 1.2 Simple approaches
   - 1.3 The density operator (reminder ... or not)
2. Master equation : The statistical physics approach
   - 2.1 Formal derivation of the master equation
   - 2.2 Statistical equilibrium : The damped harmonic oscillator
   - 2.3 Driven systems : The Optical Bloch equations
3. The quantum information approach
   - 3.1 Krauss operators
   - 3.2 Lindblad form of the master equation
   - 3.2 Quantum jumps and stochastic wavefunctions
4. Decoherence, entanglement, and control

Some References

L. Sanchez-Palencia and P. Zoller, *Lecture notes of the 2019 Les Houches summer school* (on arXiv in 202X, with X~0)

C. Gardiner and P. Zoller, *Quantum noise* (Springer-Verlag, Berlin, 2004)

Le Bellac, *Quantum physics* (Cambridge Univ Press, UK, 2012)

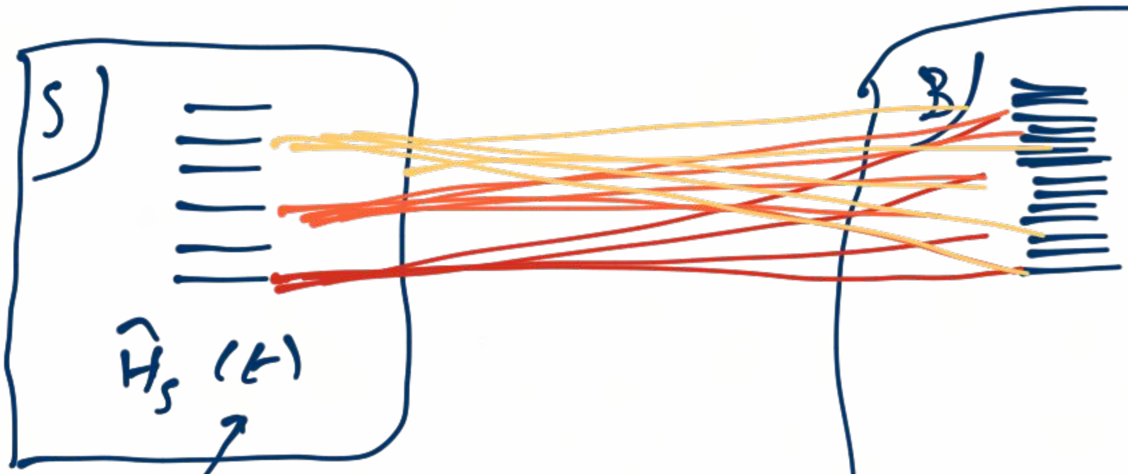
C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-photon interactions: Basic processes and applications*, (Wiley & sons, 1998)

S. Haroche, lectures at Collège de France (2003-04) ; available on line at https://www.college-de-france.fr/site/serge-haroche/_course.htm

Introduction to open quantum systems

⇒ See slides #1

1. Position of the problem



(external drive)

⇒ unspecified

Coupling : $\hat H_I$

assume H_I "<<" H_S, H_B

→ Weakly affects B

→ May strongly affect S.

Bath of. Harm. oscill.

$$\hat H_B = \sum_\ell$$

$$= \sum_\ell \hbar\omega_\ell (\hat b_\ell^\dagger \hat b_\ell + 1/2)$$

$$\ell = (\omega_\ell, \dots)$$

Fock states : $|\{N_\ell, \ell \in \mathcal{N}\}\rangle$

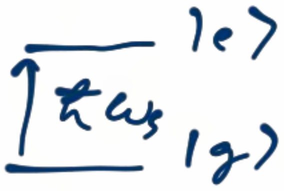
$$= \bigotimes_\ell |N_\ell\rangle$$

$$\begin{cases} \hat b_\ell^+ |N_\ell\rangle = \sqrt{N_\ell + 1}\,|N_\ell + 1\rangle \\ \hat b_\ell |N_\ell\rangle = \sqrt{N_\ell}\,|N_\ell - 1\rangle \end{cases}$$

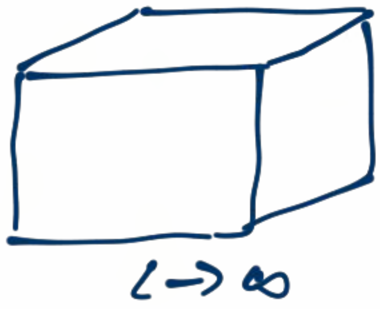
⇒ Huge (is a quasi-continuum)

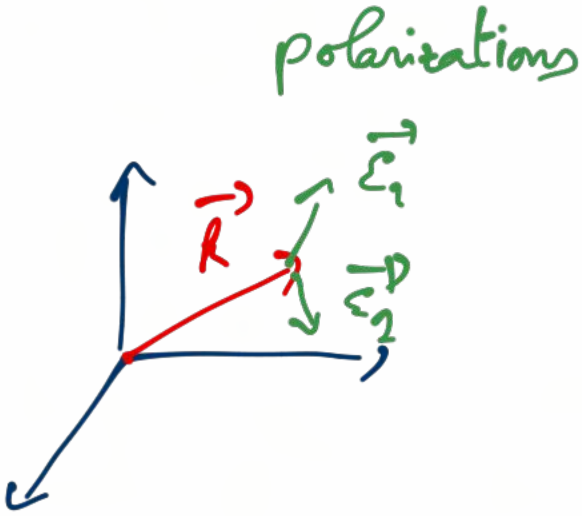
Simple example: An atom coupled to the electro-magnetic (E-M) field

- System S



- Bath B : Quantized E-M.





$k_x \in \frac{2\pi}{L}\mathbb{Z}$

$\Delta k_x \propto 1/L \to 0$

(quasi-continuum)

$\omega_\ell = c|\vec{k}|$

$\ell \equiv (\omega, \vec{\varepsilon})$

- Coupling : Dipolar approx.

$$H_I = -\vec{D}_{A/S} \cdot \hat{\vec{E}}(\vec{R})$$

atom

$$\vec{D} = \vec{d}\,|e\rangle\langle g| + \vec{d}^{\,*}\,|g\rangle\langle e|$$

field

$$\vec{E}(\vec{R}) = \sum_\ell \Big[i\mathcal{E}_\ell\, \vec{\varepsilon}_\ell\, e^{i\vec{k}_\ell\cdot\vec{R}}\, \hat{b}_\ell + H.c. \Big]$$

Rotating wave approx. (RWA):

$$\begin{aligned}\hat{H}_I = \sum_\ell \Big[& (\qquad) |e\rangle\langle g| \hat{b}_\ell \\ & + (\qquad) \cancel{|e\rangle\langle g| \hat{b}_\ell^\dagger} \\ & + (\qquad) \cancel{|g\rangle\langle e| \hat{b}_\ell} \\ & + (\qquad) |e\rangle\langle g| \hat{b}_\ell^\dagger \Big]\end{aligned}$$

2. Wavefunction approaches

Write $|\Psi(t)\rangle_{S\otimes B}$, initially $|i\rangle$

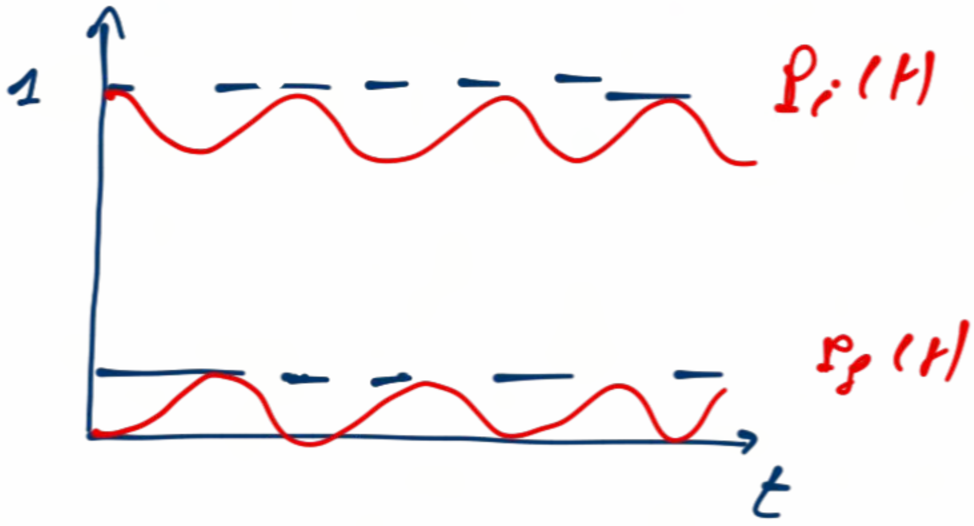
coupling to $|f\rangle$?.

2.1 Fermi golden rule

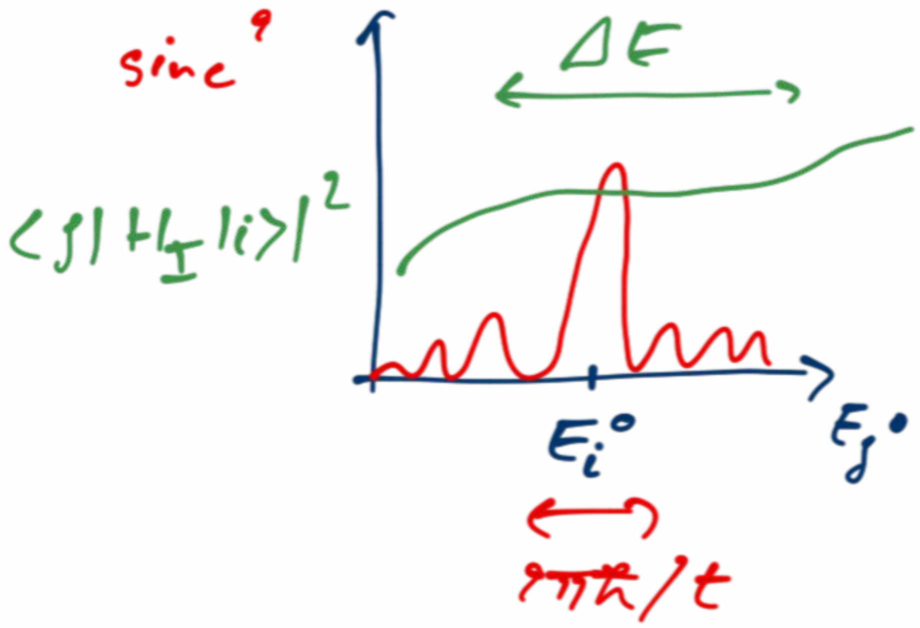
- ⊛ $|i\rangle$ coupled to a continuum of $|f\rangle$.
- ⊛ $$|\Psi(t)\rangle = \sum_j \underbrace{e^{-iE_j^0 t/\hbar}}_{\text{interaction picture.}} \overset{\alpha_j(0)=\delta_{ji}}{\alpha_j(t)} \underbrace{|j\rangle}_{\text{eigenstates of } \hat{H}_0 = H_S + H_B}$$
- ⊛ Schrödinger eq → pert. theory $(H_I \ll H_{S/B})$

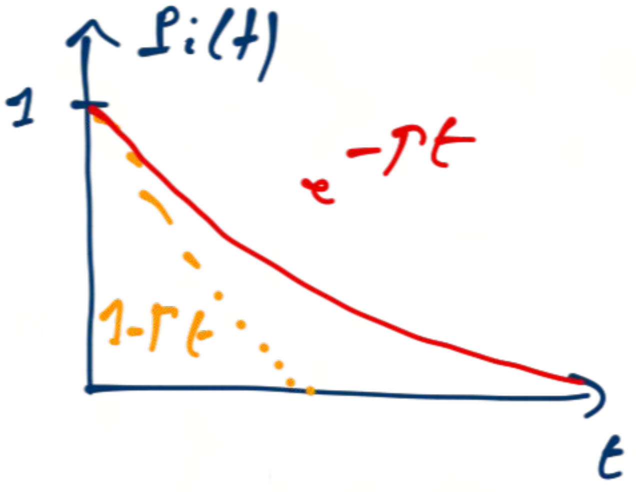
$$\Rightarrow \boxed{W_{i\to f} = \frac{|\langle f|H_I|i\rangle|^2}{\hbar^2}\, t^2 \operatorname{sinc}^2\left[\frac{(E_f^0 - E_i^0)t}{2\hbar}\right]}$$

case 1: A unique $|f\rangle$: Rabi oscillations.



case 2: coupling to a continuum





$$\operatorname{sinc}^2(\ ) \longrightarrow \propto \frac{1}{t}\,\delta(E_f^0 - E_i^0)$$

$\rightarrow$ valid for $\Delta E \gg 2\pi\hbar/t$

$$\Rightarrow \quad \rho_i(t) = 1 - \sum_{f\neq i} W_{i\to f}(t) = 1 - \Gamma t$$

$$\boxed{\begin{aligned}\Gamma &= \sum_f \frac{2\pi}{\hbar} |\langle f | H_I | i \rangle|^2 \delta(E_f^0 - E_i^0) \\ &\simeq \frac{2\pi}{\hbar} |\langle f | H_I | i \rangle|^2 \mathcal{N}_{S\otimes B}(E_i^0)\end{aligned}}$$

Density of states of $S\otimes B$

Remarks: ① Energy selection (time).

② continuum is crucial

③ Energy / time scales.

$$\boxed{\frac{2\pi\hbar}{\Delta E} \ll t \ll \frac{1}{\Gamma}}$$

④ linear decay is an artifact of pert. th

Neglecting coherence ⇒ classical rate equation

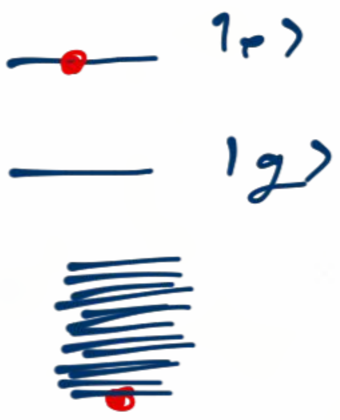
$$dP_i(t) = -\sum_{f\neq i} dW_{i\to f}\, P_i(t) = -\Gamma dt\, P_i$$

$$\Rightarrow \quad \boxed{P_i(t) = e^{-\Gamma t}}$$

2.2 Weisskopf-Wigner approach

→ non perturbative

→ { Two-level syst.
Bath, initially in $|0\rangle$



- coupling

$$\hat{H}_I = \sum_\ell \left(\hbar g_\ell |e\rangle\langle g| \hat{b}_\ell + H.c.\right)$$

Dynamics: $|\Psi(t)\rangle = \alpha(t) e^{-i\omega_s t} |e, 0\rangle$

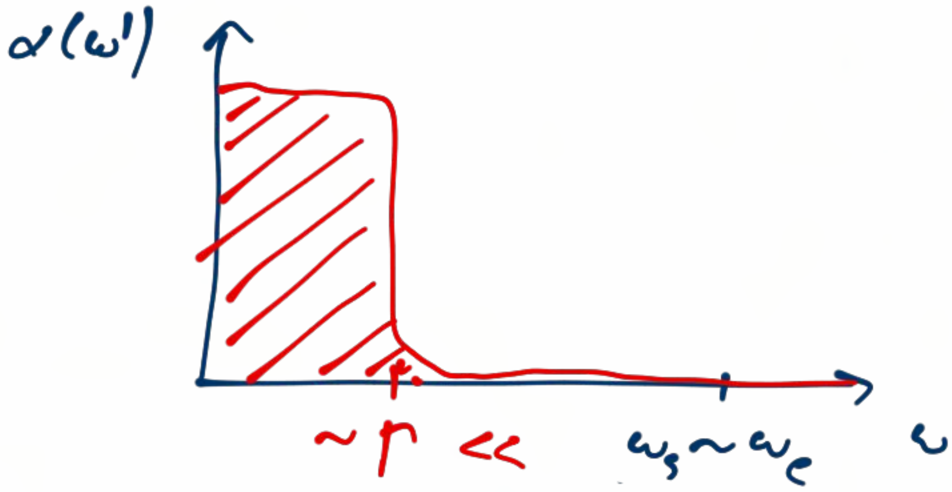
$$+ \sum_\ell \beta_\ell(t) e^{-i\omega_\ell t} |g, 1_\ell\rangle$$

Schrödinger eq:

$$\boxed{\dot{\alpha}(t) = -\sum_\ell |g_\ell|^2 \int_0^t d\tau\, \alpha(t-\tau) e^{-i(\omega_\ell-\omega_s)\tau}}$$

$\Rightarrow$ Integro-diff. equation (hard to solve)

Weisskopf-Wigner (Markov) approx:



Hyp: H_I is weak.

$$\boxed{\dot{\alpha}(t) \simeq \alpha(t) \sum_\ell |g_\ell|^2 \left[\pi \delta(\omega_s - \omega_\ell) + i \mathcal{P}\left(\frac{1}{\omega_\ell - \omega_s}\right)\right]}$$

$\Rightarrow$ Markov.

$$\boxed{\dot{\alpha}(t) = -\left(\frac{\Gamma}{2} + i\Delta\right)\alpha(t)}$$

$$\Rightarrow \begin{cases} \alpha(t) = \alpha(0)\, e^{-(\Gamma/2 + i\Delta)t} \\ P_i(t) = |\alpha(t)|^2 = e^{-\Gamma t} \end{cases}$$

Remarks: (1) Markov process

(2) Separate $\begin{cases} \text{evol. } H_0 = H_S + H_B \\ \text{evol. } H_I. \end{cases}$

↳ via the interaction picture

(3) We have recovered the exp. decay of $|i\rangle$ (consistent with neglecting the coherences, see §2.2)

3. The density operator

⇒ See slides #2

Description of a Non-Isolated System

In general, a non-isolated system S cannot be described by a ket $|\Psi_S\rangle$

Example : An atom coupled to the vacuum (spontaneous emission)

$$|\Psi(0)\rangle = |e, \vec{p}=0\rangle_A \otimes |0\rangle_R$$

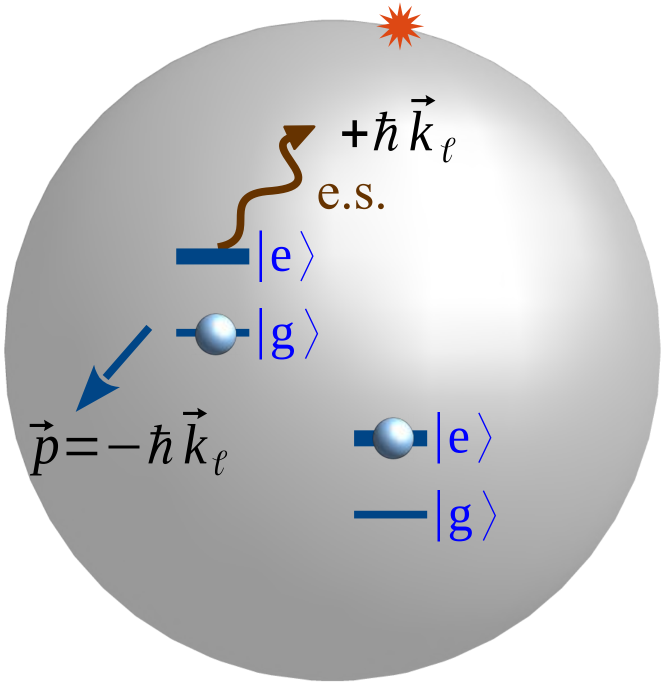
atom-radiation field coupling

$$|\Psi(t)\rangle = \alpha_0(t)|e, \vec{p}=0\rangle_A \otimes |0\rangle_R + \sum_\ell \beta_\ell(t)|g, \vec{p}=-\hbar\vec{k}_\ell\rangle_A \otimes |1\rangle_\ell$$

« measurement » (absorption of a photon)

$|\Psi'\rangle = |g, \vec{p}=-\hbar\vec{k}_\ell\rangle_A \otimes |1\rangle_\ell$ with probability $\Pi_\ell \propto |\beta_\ell|^2$

$|\psi'\rangle_A = |g, \vec{p}=-\hbar\vec{k}_\ell\rangle_A$ with probability $\Pi_\ell \propto |\beta_\ell|^2$



⇒ The system is « prepared » in a non-well defined state

The Statistical Physics Picture

Density operator

Assume that the system S is prepared in the ket $|\Psi_n\rangle$ with probability Π_n.

The average value of an observable $\hat{O}$ reads as $\langle O\rangle = \sum_n \Pi_n \langle \Psi_n|\hat{O}|\Psi_n\rangle$.

⇒ The « state » of the system is thus defined by the ensemble $\{\Pi_n, |\Psi_n\rangle\}$.

To merge all the information about the « state » of S in a single quantity, write

$$\langle O\rangle = \sum_n \Pi_n \langle \Psi_n|\hat{O}|\Psi_n\rangle = \sum_n \sum_\alpha \Pi_n \langle \Psi_n|\hat{O}|\alpha\rangle\langle\alpha|\Psi_n\rangle = \sum_\alpha \sum_n \langle\alpha|\Psi_n\rangle \Pi_n \langle\Psi_n|\hat{O}|\alpha\rangle$$

$$= \sum_\alpha \langle\alpha|\left(\sum_n \Pi_n |\Psi_n\rangle\langle\Psi_n|\hat{O}\right)|\alpha\rangle$$

↑

$\mathrm{Id} = \sum_\alpha |\alpha\rangle\langle\alpha|$

The state of a non-isolated system is fully determined by the density operator

$$\hat{\rho} = \sum_n \Pi_n |\Psi_n\rangle\langle\Psi_n|$$

The average value of an observable $\hat{O}$ then reads as $\langle O\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{O}\right)$.

<u>N.B. :</u> We have not assumed that the $|\Psi_n\rangle$'s are orthogonal.

The Statistical Physics Picture

Density operator $\hat{\rho}=\sum_n \Pi_n |\Psi_n\rangle\langle\Psi_n|$

Immediate properties :

(i) $\hat{\rho}$ is a positive Hermitian operator (in general non definite)

(ii) Normalization : $\boxed{\mathrm{Tr}(\hat{\rho})=1}$

(iii) $\hat{\rho}$ can be diagonalized in an orthonormal basis $\{|\alpha\rangle\}$:

$$\hat{\rho}=\sum_\alpha \Pi_\alpha |\alpha\rangle\langle\alpha|$$

$\Rightarrow$ The eigenvalues of $\hat{\rho}$, $0\leqslant\Pi_\alpha\leqslant 1$, may be interpreted as the probabilities that the system is in $|\alpha\rangle$.

<u>Consequence :</u> Any positive Hermitian operator, with a unit trace can be as the density operator of a given system in a certain state.

The Statistical Physics Picture

When can a system be described by a ket ?

(*) For an isolated system described by $|\Psi\rangle$, we have $\hat{\rho}=|\Psi\rangle\langle\Psi|$ (projector on $|\Psi\rangle$). Hence, $\hat{\rho}=\hat{\rho}^2$ and

$$\mathrm{Tr}\left(\hat{\rho}^2\right)=1$$

(*) In general, we may write $\hat{\rho}=\sum_\alpha \Pi_\alpha|\alpha\rangle\langle\alpha|$ with $\{|\alpha\rangle\}$ an orthonormal basis, and we have

$$\mathrm{Tr}\left(\hat{\rho}^2\right)=\sum_\alpha \Pi_\alpha^2\leqslant\sum_\alpha \Pi_\alpha=1$$

Hence, $\mathrm{Tr}\left(\hat{\rho}^2\right)\leqslant 1$, with equality iff $\hat{\rho}$ has a single non-zero eigenvalue (*e.g.* $\Pi_0=1$ and $\Pi_{\alpha>0}=0$).

A system can be described by a ket, *i.e.* $\hat{\rho}=|\Psi\rangle\langle\Psi|$

iff its density operator $\hat{\rho}$ has a single non-zero eigenvalue

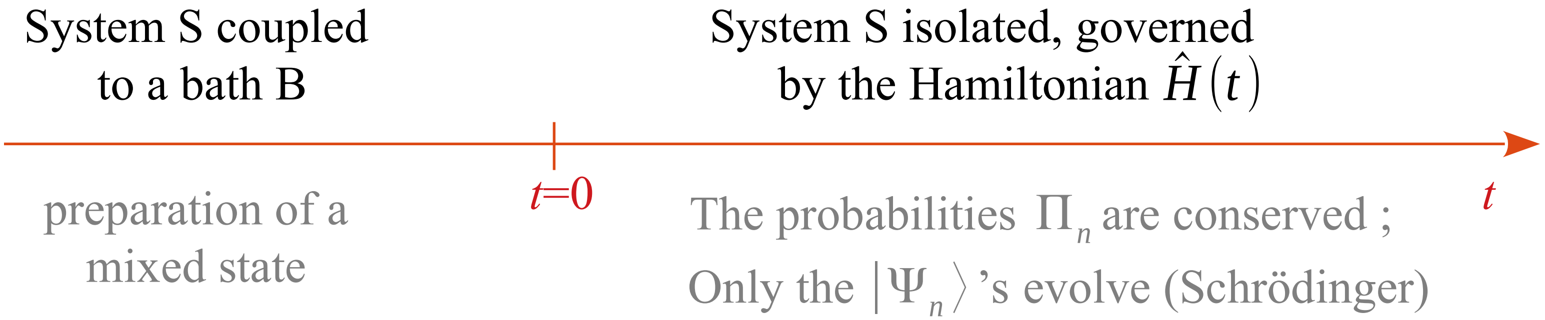
iff $\mathrm{Tr}\left(\hat{\rho}^2\right)=1$ (purity)

We then call it a pure state. Otherwise, it is called a mixed state.

The Statistical Physics Picture

Quantum Liouville equation

Assume that S is prepared in a mixed state, then isolated, and let is evolve.



System S coupled
to a bath B

System S isolated, governed
by the Hamiltonian $\hat{H}(t)$

preparation of a
mixed state

The probabilities Π_n are conserved ;
Only the $|\Psi_n\rangle$'s evolve (Schrödinger)

The dynamical equation follows :

$$i\hbar\frac{d\hat{\rho}}{dt}=i\hbar\frac{d}{dt}\left(\sum_n \Pi_n|\Psi_n\rangle\langle\Psi_n|\right)=\sum_n \Pi_n\left(i\hbar\frac{d|\Psi_n\rangle}{dt}\langle\Psi_n|+i\hbar|\Psi_n\rangle\frac{d\langle\Psi_n|}{dt}\right)$$

$$=\sum_n \Pi_n\left(\hat{H}(t)|\Psi_n\rangle\langle\Psi_n|-|\Psi_n\rangle\langle\Psi_n|\hat{H}(t)\right)=\hat{H}(t)\hat{\rho}-\hat{\rho}\hat{H}(t)$$

The dynamical equation for a system isolated during the evolution reads as

$$\frac{d\hat{\rho}}{dt}=\frac{1}{i\hbar}\left[\hat{H}(t),\hat{\rho}\right]$$

It is the *quantum Liouville equation*. Its solution reads as $\hat{\rho}(t)=U(t)\hat{\rho}(0)U^{\dagger}(t)$.

The Entanglement Picture

Let us now take into account the coupling of S with the bath B. The system S⊗B is isolated.

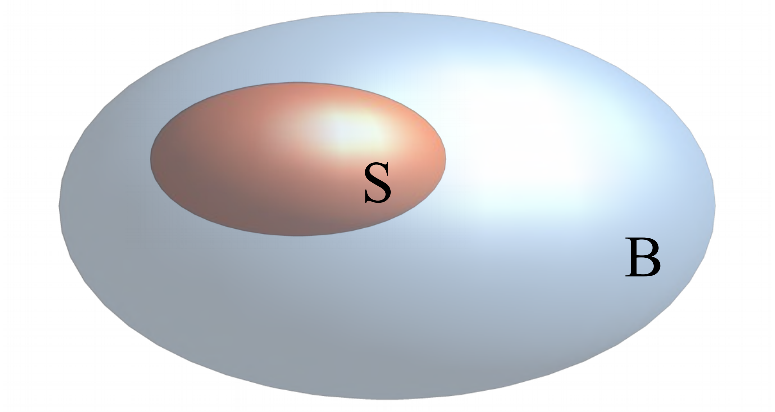
The most general pure state of S⊗R reads as

$$\boxed{|\Psi\rangle_{S\otimes B} = \sum_{1\leqslant n\leqslant \dim(\mathcal{H}_B)} c_n |\psi_n\rangle_S \otimes |\chi_n\rangle_B}$$

$\sum_n |c_n|^2 = 1$ (normalization of $|\Psi\rangle_{S\otimes B}$)

orthonormal basis of $\mathcal{H}_B$.

Normalized states of $\mathcal{H}_S$,
possibly nonorthogonal



In general, the pure state $|\Psi\rangle_{S\otimes B}$ is an entangled state of the subsystems S and B.

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« measurement » (absorption of a photon)

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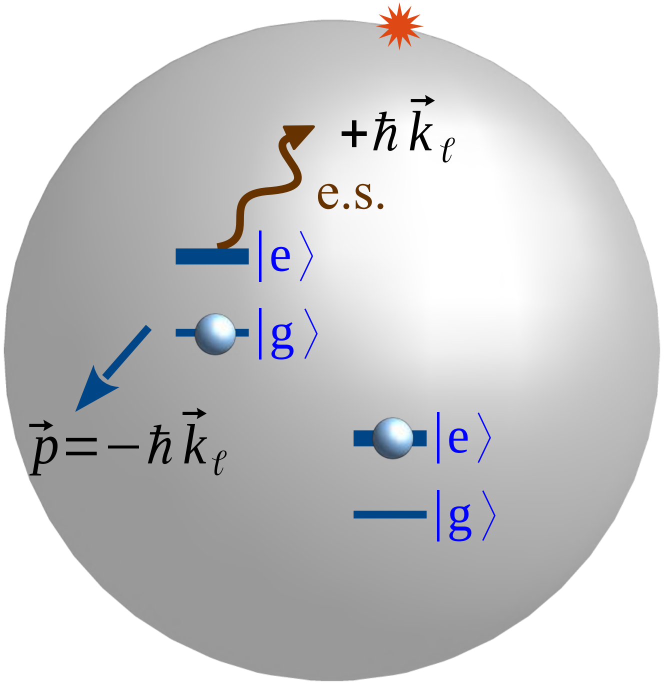

⇒ The system is « prepared » in a non-well defined state

The Entanglement Picture

Consider S⊗B as composed of the two subsystems S and B.

Example : An atom coupled to the vacuum (spontaneous emission)

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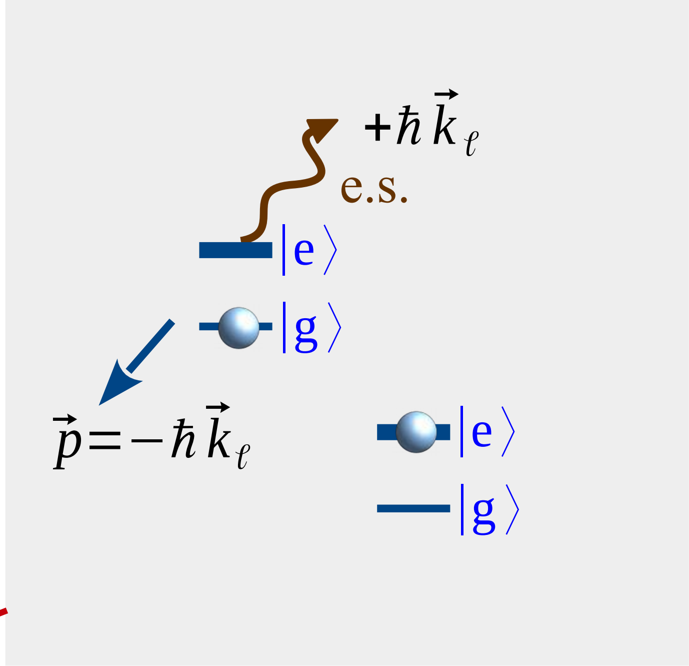
~~« measurement » (absorption of a photon)~~

~~$|\Psi'\rangle = |g, \vec{p}=-\hbar\vec{k}_\ell\rangle_A \otimes |1\rangle_\ell$ with probability $\Pi_\ell \propto |\beta_\ell|^2$~~

~~$|\psi'\rangle_A = |g, \vec{p}=-\hbar\vec{k}_\ell\rangle_A$ with probability $\Pi_\ell \propto |\beta_\ell|^2$~~



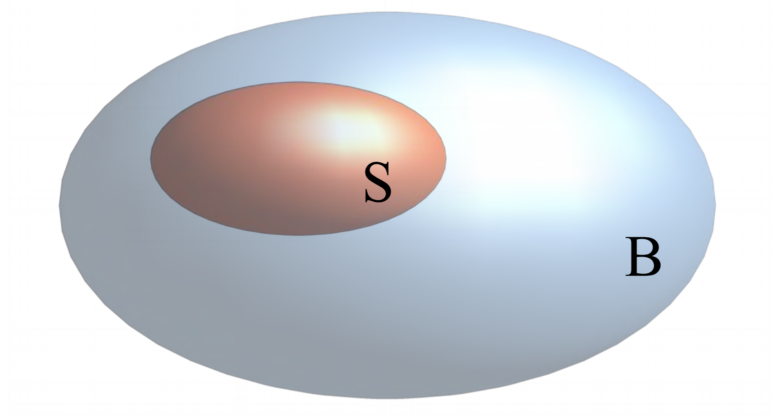
⇒ Can we describe the state of *S* , which is entangled which that of *B* with a density operator ?

The Entanglement Picture

As before, we need be able to determine the results of any measurement on S alone (independently of B).

Let $\hat{O}_{\mathrm{S}}$ be observable on S alone. For the state

$$|\Psi\rangle_{\mathrm{S}\otimes\mathrm{B}} = \sum_{1\leqslant n\leqslant \dim(\mathscr{H}_{\mathrm{B}})} c_n |\psi_n\rangle_{\mathrm{S}} \otimes |\chi_n\rangle_{\mathrm{B}}$$



we have $\langle O_{\mathrm{S}}\rangle = {}_{\mathrm{S}\otimes\mathrm{B}}\langle \Psi | \left(\hat{O}_{\mathrm{S}} \otimes \hat{1}_{\mathrm{B}}\right) | \Psi\rangle_{\mathrm{S}\otimes\mathrm{B}}$

$$= \sum_n |c_n|^2 {}_{\mathrm{S}}\langle \psi_n | \hat{O}_{\mathrm{S}} | \psi_n\rangle_{\mathrm{S}}$$

→ We find the same formula as when S is in the ket $|\psi_n\rangle$ avec probabilité $\Pi_n \equiv |c_n|^2$!

The state of the subsystem *S* is described by the density operator $\hat{\rho}_{\mathrm{S}} = \sum_n |c_n|^2 |\psi_n\rangle\langle\psi_n|$.

The average of an observable $\hat{O}_{\mathrm{S}}$ on S alone reads as $\langle O_{\mathrm{S}}\rangle = \mathrm{Tr}\left(\hat{\rho}_{\mathrm{S}} \hat{O}_{\mathrm{S}}\right)$.

N. B. : The quantity $\hat{\rho}_{\mathrm{S}}$ fulfills all the properties of a density operator (Hermitian, positive, unit trace)

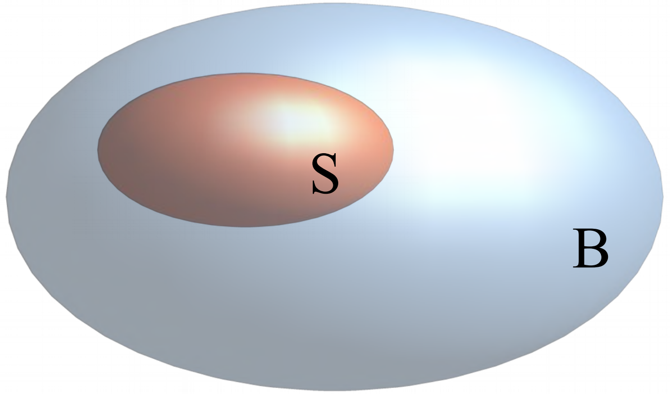
The Entanglement Picture

This result also applies to the case where S⊗B is itself in a mixed state, described by the density operator $\hat{\rho}_{S\otimes B}$.

For an observable $\hat{O}_S$ acting on S alone, one can split the traces acting on different Hilbert spaces. One then finds

$$\langle O_S \rangle = \mathrm{Tr}_{S\otimes B}\left(\hat{\rho}_{S\otimes B}\hat{O}_S\right) = \mathrm{Tr}_S\left(\hat{\rho}_S\hat{O}_S\right)$$

où $\hat{\rho}_S = \mathrm{Tr}_B\left(\hat{\rho}_{S\otimes B}\right)$



N. B. : It is easily shown that $\hat{\rho}_S$ fulfills all the properties of a density operator (Hermitian, positive, unit trace).

The state of the subsystem S of the system S⊗B in the mixed state described by the density operator $\hat{\rho}_{S\otimes B}$ is fully described by the reduced density operator

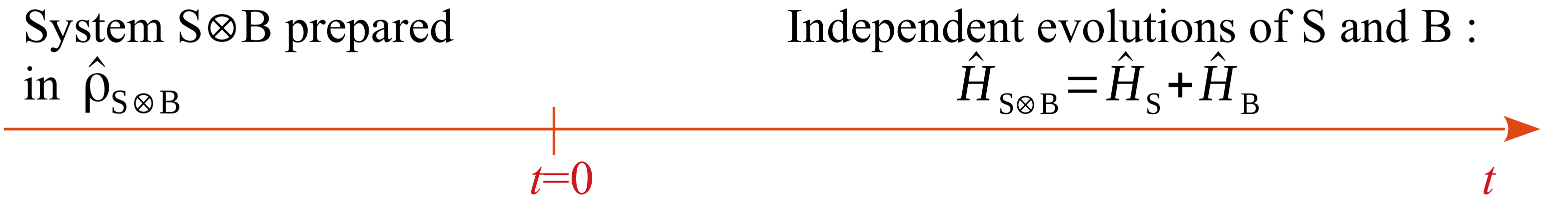
$$\hat{\rho}_S = \mathrm{Tr}_R\left(\hat{\rho}_{S\otimes B}\right) \quad \text{(partial trace)}$$

The average of an observable $\hat{O}_S$ on S alone reads as $\langle O_S \rangle = \mathrm{Tr}_S\left(\hat{\rho}_S\hat{O}_S\right)$.

The Entanglement Picture

Quantum Liouville equation

Assume that the system S⊗B is prepared in an arbitrary mixed state, that S and B are then decoupled at *t*=0, and let the system evolve.



System S⊗B prepared
in $\hat{\rho}_{S\otimes B}$

Independent evolutions of S and B :
$\hat{H}_{S\otimes B}=\hat{H}_S+\hat{H}_B$

The dynamical equation for S⊗B reads as

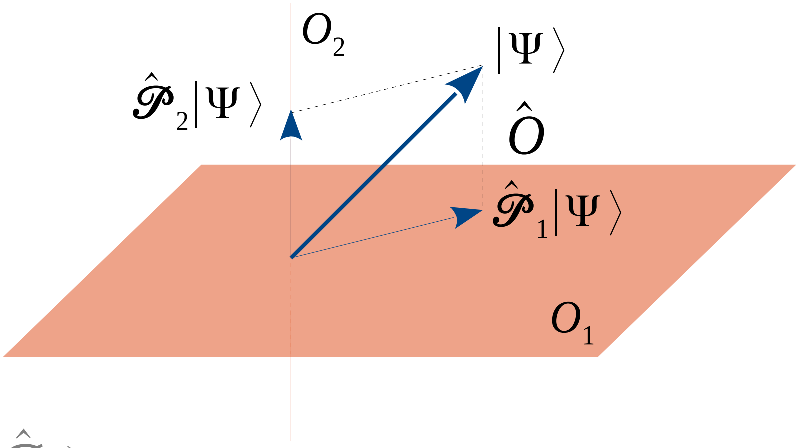
$$\begin{tikzcd}
 & i\hbar\frac{d\hat{\rho}_{S\otimes B}}{dt}=\left[\hat{H}_S+\hat{H}_B,\hat{\rho}_{S\otimes B}\right] \arrow[dl, "\mathrm{Tr}_B"'] \arrow[dr, "\mathrm{Tr}_S"] & \\
i\hbar\frac{d\hat{\rho}_S}{dt}=\left[\hat{H}_S,\hat{\rho}_S\right] & & i\hbar\frac{d\hat{\rho}_B}{dt}=\left[\hat{H}_B,\hat{\rho}_R\right]
\end{tikzcd}$$

The dynamical equation of a subsystem isolated during the evolution reads as

$$\frac{d\hat{\rho}_S}{dt}=\frac{1}{i\hbar}\left[\hat{H}_S(t),\hat{\rho}_S\right]$$

It is the *quantum Liouville equation*. Its solution reads as $\hat{\rho}_S(t)=U_S(t)\hat{\rho}_S(0)U_S^{\dagger}(t)$.

Generalized von Neumann Measurement Scheme



Quantum measurement postulate

Assume S is in the mixed state $\hat{\rho}_S$.

Measure the observable $\hat{O}=\sum_j O_j \hat{\mathscr{P}}_j$

eigenvalues

projectors onto the eigenspaces ($\hat{\mathscr{P}}_j^2=\hat{\mathscr{P}}_j$)

If the result of the measurement is read and yields O_j

O_j is found with probability $P_j=\mathrm{Tr}\left(\hat{\rho}_S\hat{\mathscr{P}}_j\right)$

After the measurement, the state of S becomes $\hat{\rho}_S{}'=\hat{\rho}_{S|j}=\dfrac{\hat{\mathscr{P}}_j\hat{\rho}_S\hat{\mathscr{P}}_j}{\mathrm{Tr}\left(\hat{\mathscr{P}}_j\hat{\rho}_S\hat{\mathscr{P}}_j\right)}=\dfrac{\hat{\mathscr{P}}_j\hat{\rho}_S\hat{\mathscr{P}}_j}{P_j}$

→ Any further measurement yields the same result, O_j.

If the result of the measurement is not read, $\hat{\rho}_S{}'=\sum_j P_j\hat{\rho}_{S|j}=\sum_j \hat{\mathscr{P}}_j\hat{\rho}_S\hat{\mathscr{P}}_j$

→ Destruction of the coherences between the states corresponding to different values of O_j (included in the measurement postulate)