Colloque Alain Bouyssy 2021

Orsay, 16th December

Valentina Ros

Complex systems: random high-dimensional landscapes and their geometry

valentina.ros@universite-paris-saclay.fr

In these ten minutes:

Who

What

Why

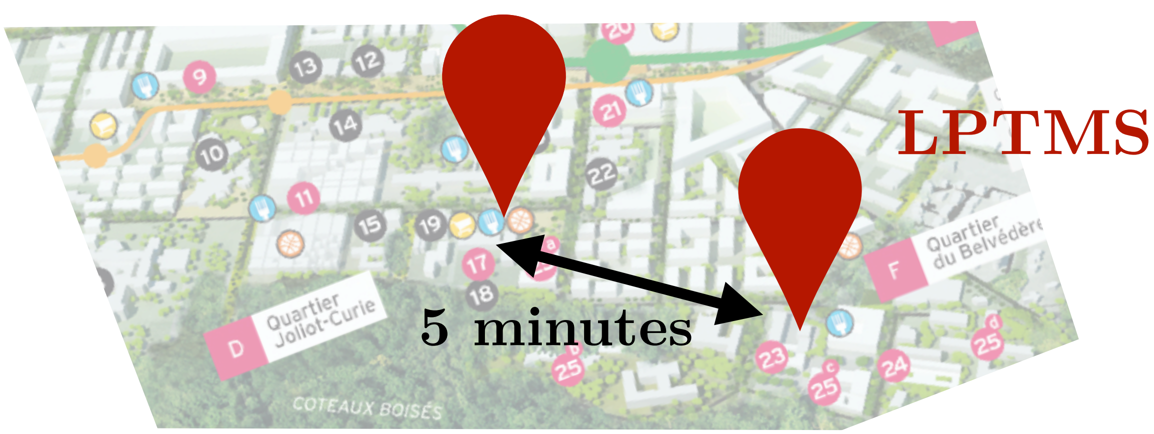
How

Who : researcher at LPTMS

Where I work:

From November 2020: working at the **Laboratoire de Physique Théorique et Modèles Statistiques** (LPTMS Orsay) as a CRCN - Chargé de Recherche de Classe Normale of CNRS.

we are here!

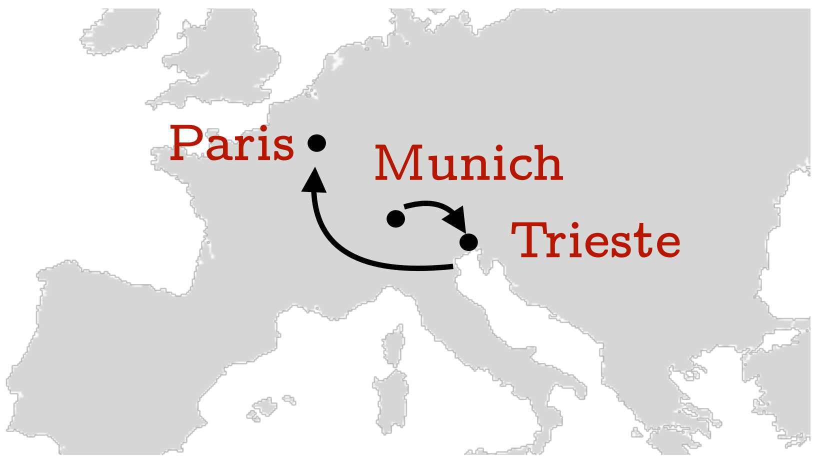


Before getting here:

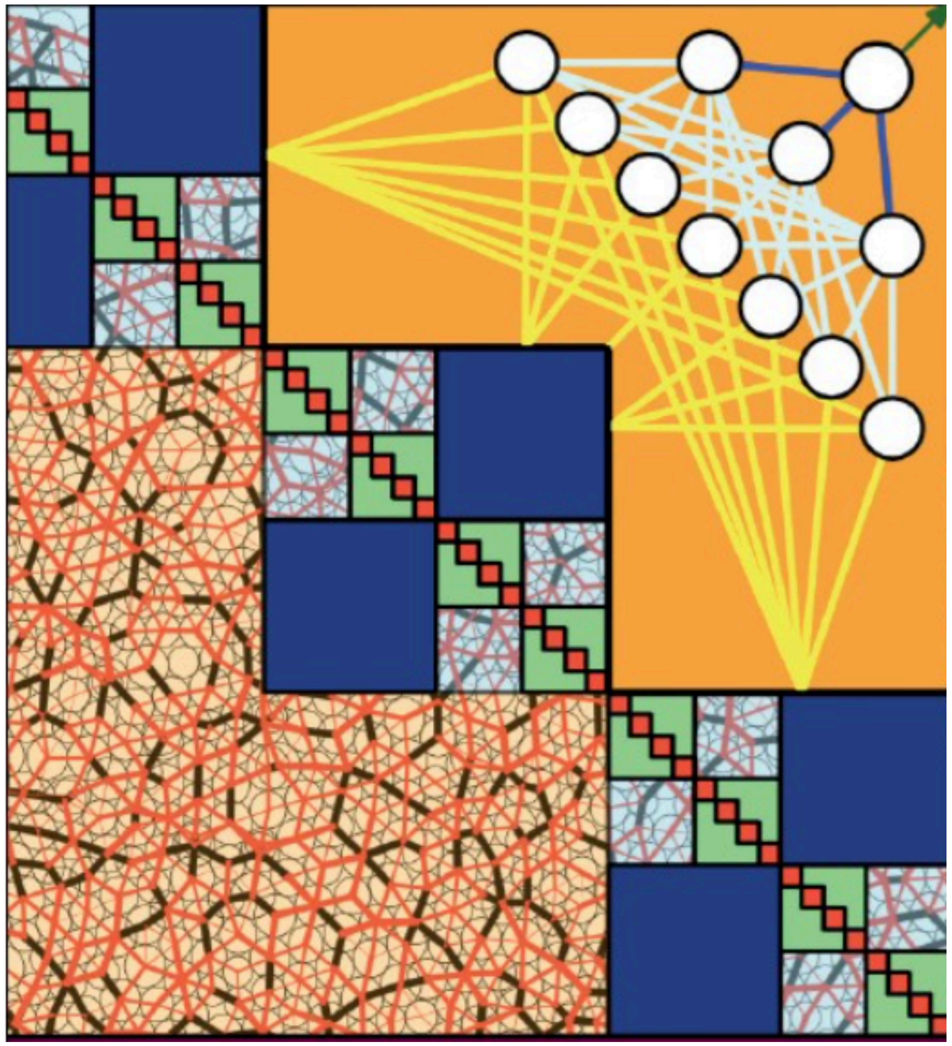
2012: Master in Mathematical Physics @ LMU-Munich

2016: Ph.D. in Statistical Physics @ SISSA-Trieste

16-20: Post Doc @ IPhT-Saclay and ENS Paris



What: disordered and complex systems



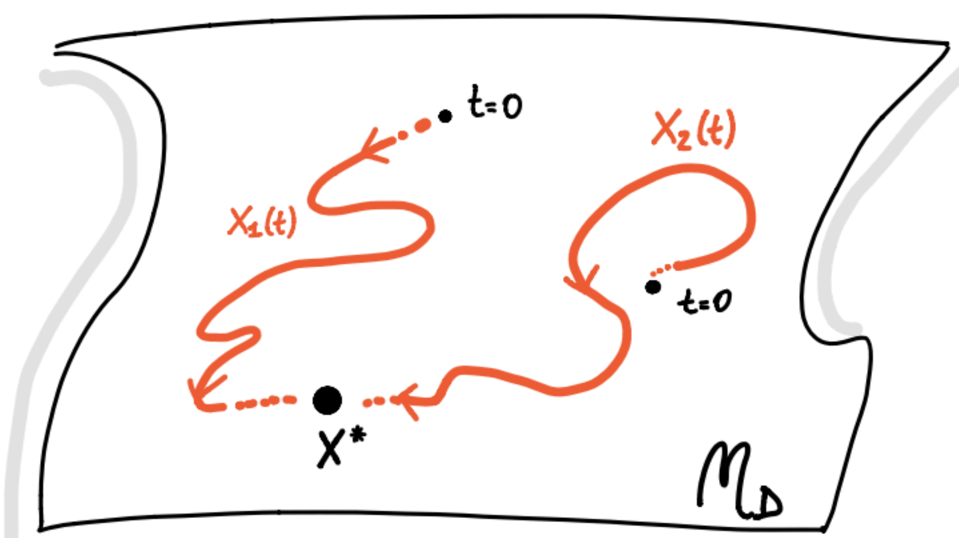
Credits Pierfrancesco URBANI
@ IPhT - Institute de Physique Theorique

- ➡ Disordered systems
- ➡ Out-of-equilibrium dynamics
- ➡ Quantum localization
- ➡ **High-dimensional landscapes**
- ➡ Glassy systems

What: counting attractors in high-D manifolds

Plenty of systems* are intrinsically high-D:
many agents interacting randomly

- Configuration: $\mathbf{x} = (x_1, \cdots, x_D) \in \mathcal{M}_D, \quad D \gg 1$
- Dynamics: $\partial_t x_i(t) = f_i(\mathbf{x}(t), \hat{a}), \quad \hat{a}$ randomness



* Ecosystems (microbiome), neural networks, financial markets, protein assemblies, glasses....

Example of special points are **equilibria $\mathbf{x}^*$:**

$$\partial_t x_i^* = f_i(\mathbf{x}^*, \hat{a}) = 0 \qquad \text{for all } i$$

They are local attractors of the dynamics.

Facts:

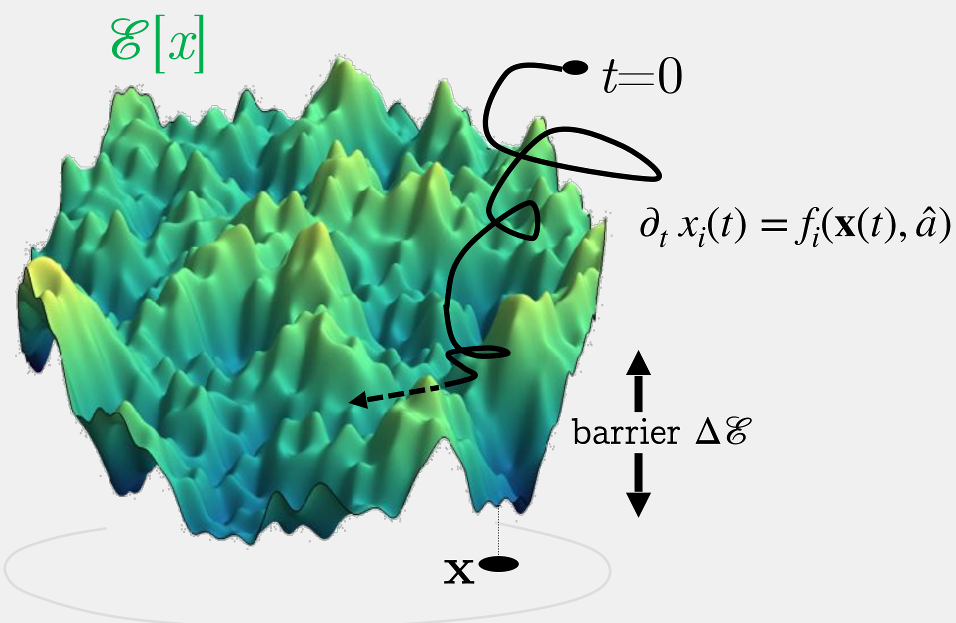
(1) High-dimensionality & random interactions tend to produce "glassiness": **huge number $\mathcal{N}_D \sim e^D$ of competing equilibria**, different from each others in terms of configurations.

(2) Dynamics with many attractors can be complex, slow, chaotic, strongly non-equilibrium.

Goal: to understand dynamics quantitatively, **count & classify** all equilibria as a function of their (linear) stability & properties.

Why: Optimizing a non-convex landscape...

Conservative problems: $\dot{x}_i = f_i(\mathbf{x}, \hat{\alpha}) = -\partial_{x_i}\mathscr{E}(\mathbf{x}, \hat{\alpha})$ is the derivative of **some high-d landscape**. Equilibria are stationary points & stability is encoded in landscape curvature



$\mathscr{M}_D$ of dimension $D \gg 1$

- How many minima, maxima, saddles at given height of landscape $\mathscr{E}$?
- What saddles connecting pairs of minima (height of barriers separating them)?

Energy landscapes in toy-models in physics

$\mathbf{x}$ conf of particles/spins, $\mathscr{M}_D$ sphere, $\sum_{i=1}^{D} x_i^2 = D$

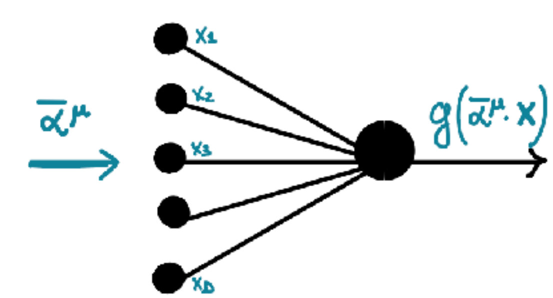
$$\mathscr{E}(\mathbf{x}, \hat{\alpha}) = \sum_{p=2}^{\infty} \sum_{i_1,\cdots,i_p} a^{(p)}_{i_1\cdots i_p} x_{i_1}\cdots x_{i_p}, \quad \text{with } a^{(p)}_{i_1\cdots i_p} \text{ random}$$

"spherical p-spin models", "random gaussian landscapes"

Loss landscapes in supervised learning

$\mathbf{x}$ = "weights"

Training set $\{\vec{\alpha}^\mu, y^\mu\}_{\mu=1}^{M}$ pairs of data-labels



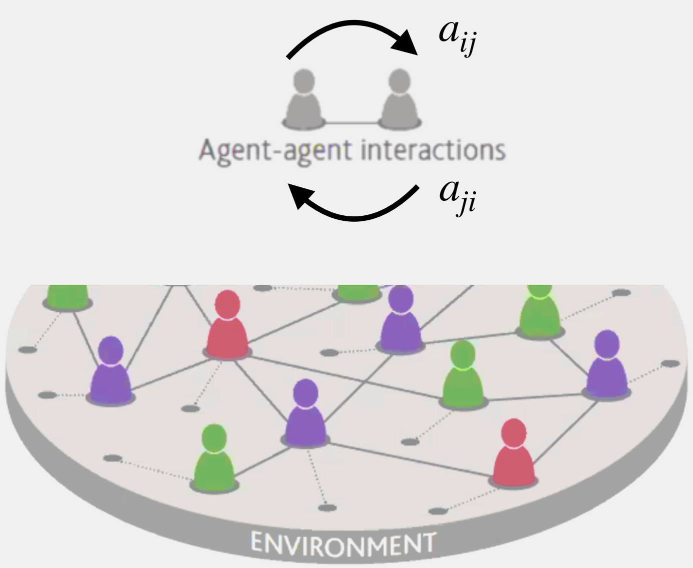
$$\mathscr{E}(\mathbf{x}, \hat{\alpha}) = \frac{1}{M}\sum_{\mu=1}^{M} \left[g(\vec{\alpha}^\mu \cdot \mathbf{x}) - y^\mu\right]^2 \quad \text{empirical loss}$$

Fitness landscapes in evolutionary biology

S. Wright 1932, R. Fisher, JBS Haldane...

... or not!

Non-conservative problems: $f_i(\mathbf{x}, \hat{\alpha})$ is a force not coming from a potential, usually because of **asymmetries in the interactions**.



- How many equilibria of given stability?
- How many with a fixed fraction of components above a certain threshold $x_i > c$?

Interacting species in theoretical ecology

$\mathbf{x}$ species abundance, $\mathcal{M}_D = \mathbb{R}^D_+$

$$f_i(\mathbf{x}, \hat{\alpha}) = x_i \left(\kappa_i - x_i - \sum_j a_{ij} x_j \right), \quad \text{with } a_{ij} \neq a_{ji}$$

"Generalized Lotka-Volterra equations" from R. May 1972

Interacting neurons in neuroscience

$\mathbf{x}$ neurons local fields, $\mathcal{M}_D = \mathbb{R}^D$

$$f_i(\mathbf{x}, \hat{\alpha}) = -x_i + \sum_j a_{ij} \phi(x_j), \quad \text{with } \phi \text{ non-linear function}$$

from Sompolinski, Crisanti, Sommers 1988

Interacting firms (or traders, or banks)

Moran, Bouchaud 2019

Sharma, Bouchaud, Tarzia, Zamponi 2020

$\mathscr{How}$: random matrices, replicas and all that

Number $\mathcal{N}_D$ of equilibria $\mathbf{x}^*$ such that $\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*, \hat{a}) = 0$ is a RV with scaling: $\mathcal{N}_D \sim e^{D\,\Sigma + o(D)}$. Goal: compute the "complexity" Σ.

The **"Kac-Rice formula"** for the moments:

$$\mathbb{E}[\mathcal{N}_D^n] = \int_{\mathcal{M}_D^{\otimes n}} \prod_{m=1}^{n} d\mathbf{x}^{(m)}\, \mathcal{P}_{\{\mathbf{x}^{(m)}\}}\left(\left\{\mathbf{f}^{(m)} = \mathbf{0}\right\}\right) \mathbb{E}_{\{\mathbf{x}^{(m)}\}}\left[\prod_{m=1}^{n} \left|\det\left(\frac{\partial f_i(\mathbf{x}^{(m)}, \hat{a})}{\partial x_j^{(m)}}\right)\right| \,\Big\|\, \left\{\mathbf{f}^{(m)} = \mathbf{0}\right\}\right]$$

Distribution of random vectors **Coupled random matrices**

A key feature of disordered systems: the distribution of $\mathcal{N}_D$ is **very broad.**
To characterise it from the moments, need to resort to **"Replica Trick"**
widely used in the theory of "glasses"

Nobel prize for physics 2021

In a nutshell

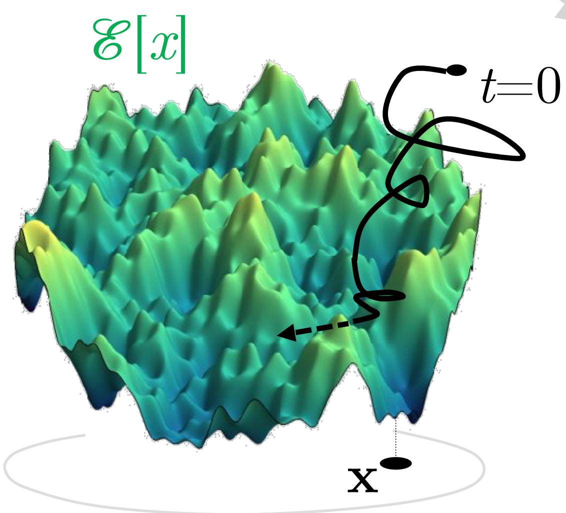
Counting equilibria in high-D manifolds is a ubiquitous problem in complex systems.
Crucial information to understand dynamics or optimization algorithms.

Topology

how many local minima
at a given energy?
Stable or unstable?

Geometry

how are minima distributed
in configuration space?
How are they connected?



configuration space
dimension $D \gg 1$

Dynamics

Which minima trap system at shorter times?
How transitions between them occur?

Thank you!

An example: glassy-to-trivial inference transitions

V. Ros et al, Physical Review X 9 (2019)

Problem: denoising of tensors. Recover information on low-rank signal in noisy tensor.

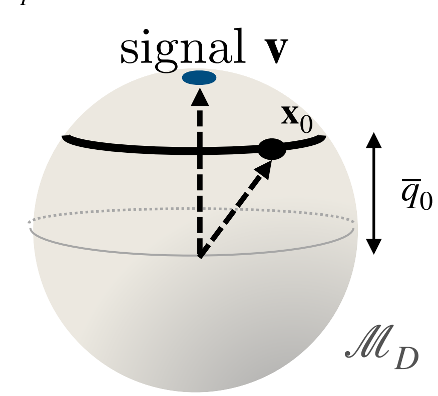
$$W_{i_1\cdots i_p} = a_{i_1\cdots i_p} - r\; v_{i_1}\cdots v_{i_p} \qquad r = \text{signal-to-noise ratio}$$

$a_{i_1\cdots i_p}$ random Gaussian, $\langle a^2_{i_1\cdots i_p}\rangle = \sigma^2/D$

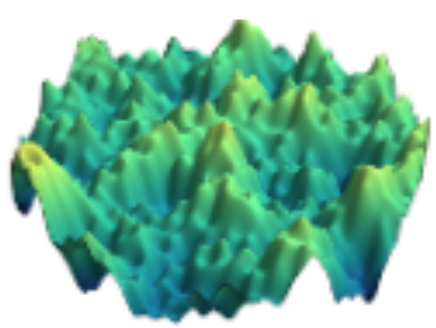
v_i deterministic signal, $\sum_{i=1}^{D} v_i^2 = 1$

"Maximum likelihood estimator" $\mathbf{x}_0$ is the ground state of the Gaussian landscape:

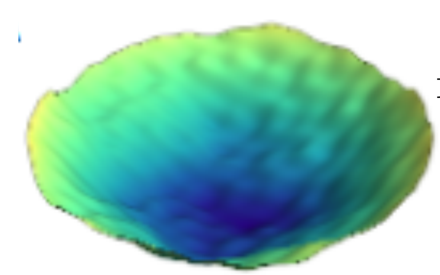
$$\mathscr{E}_{p,r}[\mathbf{x}] = \sum_{i_1<\cdots<i_p} a_{i_1\cdots i_p} x_{i_1}\cdots x_{i_p} - r\,(\mathbf{v}\cdot\mathbf{x})^p$$



many metastable minima. All uninformative



informative GS hidden by plenty of uninformative metastable minima



few minima, all informative

IMPOSSIBLE | $r_c^{(1)}$ recovery | GLASSY-HARD (only for $p > 2$) | $r_c^{(2)}$ algorithmic | EASY | r